On the Stability of a Power Control Algorithm for Wireless Networks in the presence of Time-Varying Delays

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Outline

1. Introduction
   - Wireless Ad Hoc Networks
   - Motivation
   - Power Control
2. Preliminaries
   - System Model
   - Preliminary results
   - Review of the Foschini-Miljanic Algorithm
   - FM algorithm with constant time-delays
3. Main Results
4. Conclusions
   - Summary and future work

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Wireless ad hoc networks

- **What is a wireless ad hoc network?**
  - A multi-hop wireless network
  - A self-configuring network of nodes
  - No fixed infrastructure
  - Node: source, destination or relay
- **Characteristics:**
  - No fixed infrastructure - mobility of nodes
  - Shared medium (collisions, interference)
  - Real world propagation effects
  - Distributed control systems
  - Power constraints

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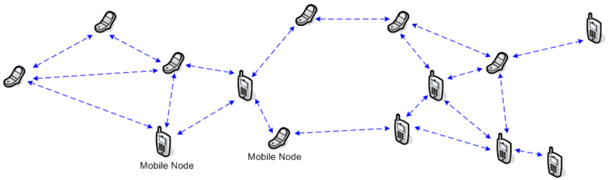
Pros and cons of wireless ad hoc networks



- **Advantages:**
  - Mobility
  - Faster speed of deployment
  - Accessibility of areas difficult to reach
  - Higher resources utilisation
  - Robustness

- **Disadvantages:**
  - Short range communications

Applications

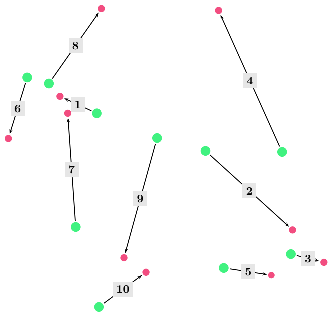
- Battlefield communications
- Disaster recovery efforts
- Impromptu communication between people
- Wireless traffic sensor networks
- Ecological habitat monitoring
- Industrial process control

Why Power Control?

- Multidimensional effect on performance determines:
  - Signal quality.
  - The range of transmission.
  - The magnitude of the interference.
- Power: a valuable resource since batteries have limited lifetime
- Power to be adjusted so that it is high enough to reach the intended receiver while causing minimal interference at other nodes

Outline

The network

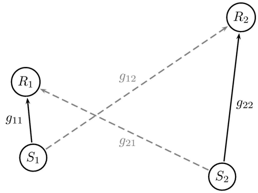


- Planar network
- Half-duplex transceivers ⇒ Unidirectional links
- Omnidirectional antennas

The channel

We consider the *Physical Model* where receivers experience **interference**:

$$I_i = \sum_{j \neq i, j \in \mathscr{T}} g_{ji} p_j + \nu. \qquad (1)$$



where

- g_{ij} the channel gain on the link between transmitter *i* and receiver *j*.
- p_i the power level chosen by transmitter *i*.
- ν the variance of thermal noise at the receiver.

The link quality is measured by the *Signal-to-Interference-and-Noise-Ratio* (SINR), given by

$$\Gamma_i = \frac{g_{ii}p_i}{\sum_{j\neq i, j\in\mathscr{T}} g_{ji}p_j + \nu} \ . \tag{2}$$

A transmission is successful (error free), if the SINR at the receiver is greater than the *capture ratio*, γ_i:

$$\frac{g_{ii}p_i}{\sum_{j\neq i, j\in\mathscr{T}} g_{ji}p_j + \nu} \geq \gamma_i \tag{3}$$

1. **When does a solution exist to such a problem?**
2. **Is it possible to be found in a distributed manner?**

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Preliminary results (1/2)

$$\frac{g_{ii}p_i}{\sum_{j\neq i, j\in\mathscr{T}} g_{ji}p_j + \nu} \geq \gamma_i$$

after manipulation, is equivalent to the following

$$p_i \geq \gamma_i \left(\sum_{j\neq i, j\in\mathscr{T}} \frac{g_{ji}}{g_{ii}} p_j + \frac{\nu}{g_{ii}} \right). \tag{4}$$

In matrix form, for a network consisting of n communication pairs, this can be written as

$$\mathbf{p} \geq C\mathbf{p} + \eta \tag{5}$$

where

$$\mathbf{p} = \begin{pmatrix} p_1 & p_2 & \dots & p_n \end{pmatrix}^T$$

$$C_{ij} = \begin{cases} 0 & \text{, if } i = j, \\ \gamma_i \frac{g_{ji}}{g_{ii}} & \text{, if } i \neq j. \end{cases}$$

$$\eta_i = \frac{\gamma_i \nu}{g_{ii}}$$

Preliminary results (2/2)

1. When does a solution exist to such a problem?

The necessary and sufficient condition for

$$\mathbf{p} \geq C\mathbf{p} + \eta$$

to have a positive solution $\mathbf{p}^*$ for a positive vector η is that the Perron-Frobenius eigenvalue of the matrix C is less than 1.

From the Perron-Frobenius theorem and standard matrix theory, the following are equivalent statements:

1. There exists a vector $\mathbf{p} > \mathbf{0}$ (i.e. $p_i > 0$ for all i) such that $(I - C)\mathbf{p} \geq \eta$
2. $(I - C)^{-1}$ exists and is positive componentwise
3. $\rho(C) < 1$.

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Review of the Foschini-Miljanic (FM) Algorithm

2. Is it possible to find the solution in a distributed manner?

The Foschini-Miljanic algorithm provably succeeds in attaining the required SINRs for all nodes in the network if a solution exists and fails if there does not exist a solution.

- The **continuous-time** algorithm ($k_i > 0$)

$$\frac{dp_i(t)}{dt} = k_i \left(-p_i(t) + \gamma_i \left(\sum_{j \neq i, j \in \mathscr{T}} \frac{g_{ji}}{g_{ii}} p_j(t) + \frac{\nu}{g_{ii}} \right) \right) \tag{6}$$

- The discrete-time algorithm ($k_i \in (0,1]$)

$$p_i(n+1) = (1-k_i)p_i(n) + k_i \gamma_i \left(\sum_{j \neq i, j \in \mathscr{T}} \frac{g_{ji}}{g_{ii}} p_j(n) + \frac{\nu}{g_{ii}} \right) \tag{7}$$

FM Algorithm with constant time-delays (1/2)

Original algorithm assumes interference measurements at the receiver are available instantaneously at the transmitter. But delays inevitably exist. Most prominent:

- Processing time (coding and decoding)
- Propagation delays
- Availability of the channel for transmission

Does the algorithm converge in the presence of constant time-delays?

The **continuous-time** algorithm now becomes

$$\frac{dp_i(t)}{dt} = k_i \left(-p_i(t) + \gamma_i \left(\sum_{j \neq i, j \in \mathscr{T}} \frac{g_{ji}}{g_{ii}} p_j(t - T_i) + \frac{\nu}{g_{ii}} \right) \right). \quad (8)$$

FM Algorithm with constant time-delays (2/2)

Theorem [Charalambous *et al*, 2008]

If the spectral radius of matrix C is less than 1, then the following power control algorithm

$$\frac{dp_i(t)}{dt} = k_i \left(-p_i(t) + \gamma_i \left(\sum_{j \neq i, j \in \mathscr{T}} \frac{g_{ji}}{g_{ii}} p_j(t - T_i) + \frac{\nu}{g_{ii}} \right) \right), \ i \in \mathscr{T} \tag{9}$$

for γ_i, g_{ji}, $\nu > 0$, is asymptotically stable for arbitrarily large delays, $T_i > 0$, for any initial state $p_i(0) > 0$ and for any proportionality constant, $k_i > 0$.

Outline

What happens in the presence of time-varying delays?

So far...

- Algorithms necessitate communication among users, hence propagation delays exist in the network.
- The Foschini-Miljanic algorithm is asymptotically stable if there are constant delays in the execution of the algorithm.

But...

- **How does the algorithm behave in the presence of time-varying delays?**
- We extend the work done in [Charalambous *et al*, 2008] in the case where the network experiences time-varying delays.

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The FM algorithm with time-varying delays

The FM algorithm when the constant time-delays are introduced is given by

$$\frac{dp_i(t)}{dt} = k_i\left(-p_i(t) + \gamma_i\left(\sum_{j\neq i, j\in\mathscr{T}} \frac{g_{ji}}{g_{ii}} p_j(t-T_i) + \frac{\nu}{g_{ii}}\right)\right). \tag{10}$$

When time-varying delays are introduced, then

$$\frac{dp_i(t)}{dt} = k_i\left(-p_i(t) + \gamma_i\left(\sum_{j\neq i, j\in\mathscr{T}} \frac{g_{ji}}{g_{ii}} p_j(t-T_i(t)) + \frac{\nu}{g_{ii}}\right)\right). \tag{11}$$

In matrix form, in the case of N nodes,

$$\frac{dp_i(t)}{dt} = k_i \left(-p_i(t) + \gamma_i \left(\sum_{j \neq i, j \in \mathscr{T}} \frac{g_{ji}}{g_{ii}} p_j(t - T_i(t)) + \frac{\nu}{g_{ii}} \right) \right).$$

can be written as

$$\frac{d\mathbf{p}(t)}{dt} = -K\mathbf{p}(t) + K \left(\sum_{k=1}^{N} A_{d_k} \mathbf{p}(t - T_k(t)) + \eta \right), \tag{12}$$

where

$$\mathbf{p}(t) = \begin{pmatrix} p_1(t) \\ \vdots \\ p_N(t) \end{pmatrix}, \qquad A_{d_{k_{ij}}} = \begin{cases} 0 & \text{, if } j = k \text{ or } i \neq k, \\ \gamma_k \frac{g_{ji}}{g_{kk}} & \text{, otherwise.} \end{cases}$$

$$K = diag(k_i), \qquad \eta = \begin{pmatrix} \gamma_1 \frac{\nu}{g_{11}} \\ \vdots \\ \gamma_N \frac{\nu}{g_{NN}} \end{pmatrix}.$$

Theorem 1

Given scalars $T_{i_m} > 0$ and $\alpha_i \geq 0$, $\forall i \in \mathscr{T}$, the system

$$\frac{d\mathbf{p}(t)}{dt} = -K\mathbf{p}(t) + K\left(\sum_{k=1}^{N} A_{d_k}\mathbf{p}(t - T_k(t)) + \eta\right),$$

is asymptotically stable for any time-varying delays $T_i(t)$ satisfying $0 \leq T_i(t) \leq T_{i_m}$, and $\dot{T}_i(t) \leq \alpha_i$, $\forall t$ and $\forall i \in \mathscr{T}$, if there exists $N \times N$ matrices $Q_k \succ 0$, $S_k \succ 0$, $R_k \succ 0$, $k = \{1, ..., N\}$ and $N \times N$ diagonal matrices $Z \succ 0$, $X \succ 0$, $Y \succ 0$ such that the following LMI holds:

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \prec 0 \text{ and } \mathrm{Z} \succeq \mathrm{W},$$

where

$$\Theta_{11} = \begin{bmatrix} T & \frac{1}{T_{1m}}R_1 + XA_{d_1} & \dots & \frac{1}{T_{Nm}}R_N + XA_{dN} \\ \frac{1}{T_{1m}}R_1 + A_{d_1}^T X & U_1 & 0 & \dots \\ \vdots & \vdots & \ddots & 0 \\ \frac{1}{T_{Nm}}R_N + A_{d_N}^T X & 0 & \dots & U_N \end{bmatrix},$$

$$\Theta_{12} = \begin{bmatrix} 0 & \dots & 0 & -Y \\ \frac{1}{T_{1m}}R_1 & 0 & \vdots & A_{d1}^T Y \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \frac{1}{T_{Nm}}R_N & A_{dN}^T Z \end{bmatrix},$$

Theorem 1 (continued)

$$\Theta_{21} = \Theta_{12}^T,$$

$$\Theta_{22} = \begin{bmatrix} V_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & V_N & 0 \\ 0 & \dots & 0 & -Z \end{bmatrix}$$

and

$$T = \sum_{k=1}^{N} \left[Q_k + S_k - \frac{1}{T_{k_m}} R_k \right] - 2X,$$

$$U_k = -(1-\alpha_k) Q_k - \frac{2}{T_{k_m}} R_k,$$

$$V_k = -S_k - \frac{1}{T_{k_m}} R_k,$$

$$W = \sum_{k=1}^{N} T_{k_m} R_k.$$

The stabilizing gain $K = diag(k_i)$ is given by $K = Z^{-1} Y$.

Sketch of the proof (1/4)

- We consider the following Lyapunov-Krasovskii functional

$$V(t,\mathbf{p}_t) = \mathbf{p}^T(t)P\mathbf{p}(t) + \sum_{k=1}^{N} \int_{t-T_k(t)}^{t} \mathbf{p}^T(\theta)Q_k\mathbf{p}(\theta)d\theta + \sum_{k=1}^{N} \int_{t-T_{km}}^{t} \mathbf{p}^T(\theta)S_k\mathbf{p}(\theta)d\theta + \sum_{k=1}^{N} \int_{t-T_{km}}^{t} \int_{\theta}^{t} \dot{\mathbf{p}}^T(u)R_k\dot{\mathbf{p}}(u)dud\theta.$$

- For some $\varepsilon > 0$, the Lyapunov-Krasovskii functional condition $V(\mathbf{p}_t) \geq \varepsilon \left\| \mathbf{p}_t(0) \right\|$ is satisfied.

Sketch of the proof (2/4)

- The derivative along the trajectories of (12) leads to

$$\dot{V}(t,\mathbf{p}_t) \leq \xi^T(t)\Gamma\xi(t) + \dot{\mathbf{p}}^T(t)W\dot{\mathbf{p}}(t),$$

where

$$\Gamma = \begin{bmatrix} T & \frac{1}{T_{1_m}}R_1 + PKA_{d1} & \cdots & \frac{1}{T_{N_m}}R_N + PKA_{dN} & 0 & \cdots & 0 \\ \frac{1}{T_{1_m}}R_1 + A_{d1}KP & U_1 & 0 & \vdots & \frac{1}{T_{1_m}}R_1 & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \ddots & \vdots \\ \frac{1}{T_{N_m}}R_N + A_{dN}^T P & 0 & \cdots & U_N & 0 & \cdots & \frac{1}{T_{N_m}}R_N \\ 0 & \frac{1}{T_{1_m}}R_1 & 0 & \vdots & V_1 & \cdots & 0 \\ \vdots & \cdots & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{T_{N_m}}R_N & 0 & \cdots & V_N \end{bmatrix}$$

Sketch of the proof (3/4)

$$\xi(t) = \begin{bmatrix} \mathbf{p}(t) \\ \mathbf{p}(t - T_1(t)) \\ \vdots \\ \mathbf{p}(t - T_N(t)) \\ \mathbf{p}(t - T_{1_m}) \\ \vdots \\ \mathbf{p}(t - T_{N_m}) \end{bmatrix}$$

$$T = \sum_{k=1}^{N} \left[Q_k + S_k - \frac{1}{T_{k_m}} R_k \right] - KP - PK,$$

$$U_k = -(1 - \alpha_k) Q_k - \frac{2}{T_{k_m}} R_k,$$

$$V_k = -S_k - \frac{1}{T_{k_m}} R_k$$

$$W = \sum_{k=1}^{N} T_{k_m} R_k.$$

Sketch of the proof (4/4)

- If we introduce a diagonal positive definite matrix Z (as decision variable) such that $Z \succeq W$, the following condition implies $\dot{V}(\mathbf{p}) \leq 0$:

$$\xi^T(t)\Gamma\xi(t) + \dot{\mathbf{p}}^T(t)Z\dot{\mathbf{p}}(t) < 0.$$

  The condition can be re-written as

$$\xi^T(t)\Gamma\xi(t) + \xi^T(t)N^T ZN\xi(t) < 0$$

  which is equivalent to

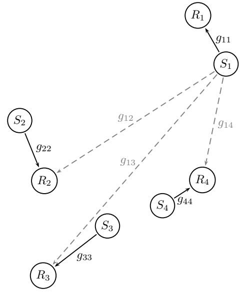
$$\Gamma + N^T ZN \prec 0 \text{ and } \mathrm{Z} \succeq \mathrm{W}$$
$$\Leftrightarrow \begin{bmatrix} \Gamma & N^T Z \\ ZN & -Z \end{bmatrix} \prec 0 \text{ and } \mathrm{Z} - \mathrm{W} \succeq 0,$$

  with $N = \begin{bmatrix} -K & KA_{d1} & \dots & KA_{dN} & 0 & \dots & 0 \end{bmatrix}$.
- Then, we perform the change of variable: $Y = ZK$ and $X = PK$.

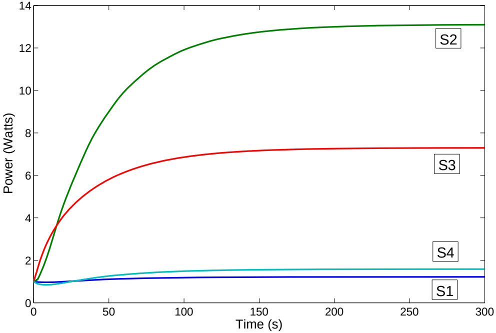
Example 1 (1/2)

We set the SINR threshold and the thermal noise for each node to $\gamma_i = 3$ and $v = 0.04$ *Watts*, respectively. The initial power $p_i(0)$ for all transmitters is set to 1 *Watt*. The network is described by matrix C_1

$$C_1 = \begin{bmatrix} 0 & 0.0163 & 0.0108 & 0.0212 \\ 0.0250 & 0 & 1.5124 & 0.2566 \\ 0.0213 & 0.2146 & 0 & 0.3564 \\ 0.0771 & 0.0111 & 0.1224 & 0 \end{bmatrix}$$



Example 1 (2/2)



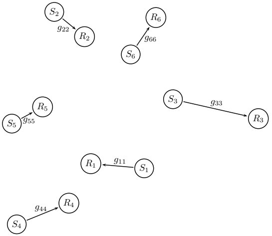
The stabilizing gain matrix (K_1) is given by

$$K_1 = diag(0.1280, 0.0608, 0.0859, 0.0919).$$

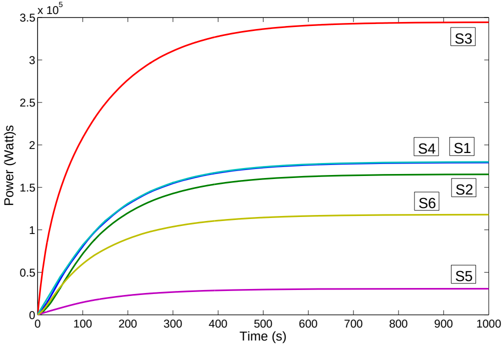
Example 2 (1/2)

We set the SINR threshold and the thermal noise for each node to $\gamma_i = 3$ and $v = 0.04$ *Watts*, respectively. The initial power $p_i(0)$ for all transmitters is set to 1 *Watt*. The network is described by matrix C_2

$$C_2 = \begin{bmatrix} 0 & 0.0414 & 0.2074 & 0.2925 & 0.3998 & 0.1345 \\ 0.0159 & 0 & 0.0506 & 0.0043 & 0.0422 & 1.164 \\ 0.7335 & 0.0626 & 0 & 0.0364 & 0.0477 & 0.4231 \\ 0.6359 & 0.0222 & 0.0644 & 0 & 0.3283 & 0.0447 \\ 0.0227 & 0.0536 & 0.0155 & 0.0215 & 0 & 0.0407 \\ 0.0228 & 0.1114 & 0.2458 & 0.0030 & 0.011 & 0 \end{bmatrix}$$



Example 2 (2/2)



The stabilizing gain matrix (K_2) is given by

$$K_2 = diag(0.0300, 0.0435, 0.0401, 0.0427, 0.0538, 0.0391).$$

Outline

Summary and future work

Summary:

- Focused on power adaptation in an environment where time-varying delays exist between communicating pairs.
- We derived the stability conditions for which the system is stable by proposing a Lyapunov-Krasovskii functional in the form of a LMI.
- Theorem 1 provides a practical and systematic condition that provides k_i ensuring the stability of the FM algorithm.
- Once these gains are embedded in each node, each power $p_i(t)$ will converge to its corresponding equilibrium point in a distributed manner.

Future work:

- Investigate if the algorithm converges regardless of the nature of the time-delays.
- Investigate how the convergence rate of the algorithm changes with time-delays.

Thank you

Questions?