Team Optimality Conditions of Differential Decision Systems with Nonclassical Information Structures

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Outline

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Motivation + Objectives: I

Motivation

Classical theory of optimization: often developed based on centralized information-classical information structures.

Decentralized decision systems: often consist of multiple observation posts and decision/control stations-nonclassical information structures.

Information structures: information available as arguments to the strategies at control stations to implement their actions.
–Applications in

1. Transportation systems, smart grid energy systems, social network systems;
2. General large scale distributed systems with local decisions.

Motivation + Objectives: II

References

Team Problems+Informations Structures+Dynamic Optimization

- **Team Theory (Cooperative)**

  [MR-72]: Marschak and R. Radner, **Economic Theory of Teams**, 1972.

  [KSM82]: J. Krainak, J.L. Speyer, and S.I. Marcus, Static Team Problems-Part I: Sufficient Conditions and the Exponential Cost Criterion, **IEEE Transactions on Automatic Control**, pp. 839–848, 1982.

  [WS]: P.R. Wall and J.H. van Schuppen, A class of Team Problems with Discrete Action Spaces: Optimality Conditions Based on Multimodularity, **SIAM Journal on Control and Optimization**, pp. 875-892, 2000.

Motivation + Objectives: III

- **Information Structures**

  [W68]: H.S. Witsenhausen, A Counterexample in Stochastic Optimum Control, **SIAM Journal on Control and Optimization**, pp.131-147, 1968.

  [W71]: H.S. Witsenhausen, Separation of Estimation and Control for Discrete Time Systems, **Proceedings of the IEEE**, pp.1557-1566, 1971.

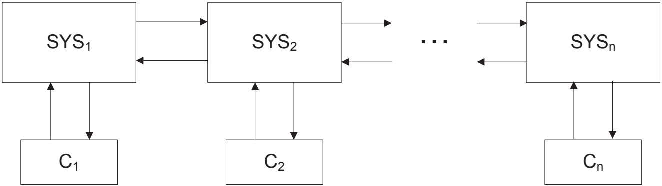
- Recent Esseys
  Jan H. van Schuppen and Tiziano Villa (editors), **Coordination Control of Distributed Systems**, Springer, 2014.

Motivation + Objectives: IV

Distributed System + Decentralized Decisions

1. Interconnected Systems
2. Decentralized decisions + Nonclassical information structures

- Example of distributed system with decentralized control



Motivation + Objectives: V

Objectives

- How do we optimize?

  1. Team optimality versus person-by-person optimality
  2. Necessary and sufficient conditions of optimality
  3. Existence of regular or randomized strategies

- Do our methods apply to other game criteria?

  1. Non-cooperative games (Nash-equilibrium)
  2. Minimax games

Outline

Team Optimality: Formulation I

- **Differential System-** $x^i \in \mathbb{R}^{n_i}, u^i \in \mathbb{R}^{d_i}$

$$\begin{aligned} \dot{x}^i(t) &= f^i(t, x^i(t), u^i_t) \\ &\quad + \sum_{j \in \mathcal{N}_i} f^{ij}(t, x^{j_1}(t), \ldots, x^{j_{|N_i|}}(t), u^{j_1}_t, \ldots, u^{j_{|N_i|}}_t), \end{aligned} \tag{1}$$

$$x^i(0) = x^i_0, \quad t \in (0, T], \quad i \in \mathbb{Z}_N \triangleq \{1, 2, \ldots, N\}$$

$\mathcal{N}_i \triangleq \{j_1, j_2, \ldots, j_{|N_i|}\}$: neighbors of subsystem i

- **Observations-** $y^i \in \mathbb{R}^{k_i}$

$$\begin{aligned} y^i(t) &\triangleq h^i(t, x^{j_1}, \ldots, x^{j_{|K_i|}}) \\ &\equiv h^i(t, \{x^{j_i}(s), \ldots, x^{j_{|K_i|}}(s) : 0 \leq s \leq t\}), \quad i \in \mathbb{Z}_N \end{aligned} \tag{2}$$

Team Optimality: Formulation II

- **Team Pay-Off**

$$
\begin{aligned}
J(u) \equiv & J(u^1, u^2, \ldots, u^N) \\
& \triangleq \int_0^T \ell(t, x^1(t), \ldots, x^N(t), u_t^1, \ldots, u_t^N) dt + \Phi(x^1(T), \ldots, x^N(T)).
\end{aligned} \tag{3}
$$

- **Compact Representation**

$$
\begin{aligned}
x &\triangleq Vector\{x^1, \ldots, x^N\}, \qquad u \triangleq Vector\{u^1, \ldots, u^N\}, \\
y &\triangleq Vector\{y^1, \ldots, u^N\}, \qquad h \triangleq Vector\{h^1, \ldots, h^N\},
\end{aligned}
$$

$$
\dot{x}(t) = f(t, x(t), u_t), \quad t \in (0, T], \tag{4}
$$

$$
y(t) = h(t, x), \qquad t \in [0, T]. \tag{5}
$$

Team Optimality: Formulation III

Preliminaries

- $\mathcal{H} = M \bigoplus M^{\perp}$: direct sum representation of a Hilbert space $\mathcal{H}$;
- $\mathbf{P}_M(x)$: orthogonal projection of a Hilbert space element $x \in \mathcal{H}$ onto the closed subspace $M \subset \mathcal{H}$.

- $C([0,T],\mathbb{R}^n) \stackrel{\triangle}{=} \Big\{ \text{continuous functions } \phi : [0,T] \longrightarrow \mathbb{R}^n : \sup_{t\in[0,T]} |\phi(t)|_{\mathbb{R}^n} < \infty \Big\}$;
- $B^{\infty}([0,T],\mathbb{R}^n) \stackrel{\triangle}{=} \Big\{ \text{measurable functions } \phi : [0,T] \longrightarrow \mathbb{R}^n : ||\phi||^2 \stackrel{\triangle}{=} \sup_{t\in[0,T]} |\phi(t)|^2_{\mathbb{R}^n} < \infty \Big\}$;
- $L^2([0,T],\mathbb{R}^n) \stackrel{\triangle}{=} \Big\{ \phi : [0,T] \longrightarrow \mathbb{R}^n : \int_{[0,T]} |\phi(t)|^2_{\mathbb{R}^n} dt < \infty \Big\}$;
- $L^2([0,T],\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)) \stackrel{\triangle}{=} \Big\{ \Sigma : [0,T] \longrightarrow \mathbb{R}^{n\times m} : \int_{[0,T]} |\Sigma(t)|^2_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)} dt \stackrel{\triangle}{=} \int_{[0,T]} tr(\Sigma^*(t)\Sigma(t)) dt < \infty \Big\}$.

Team Optimality: Formulation IV

- **Decentralized Strategies**

$$\mathbb{U}^i[0,T] \triangleq \Big\{ u^i \in L^2([0,T],\mathbb{R}^{d_i}) : \; u^i_t \in \mathbb{A}^i \subset \mathbb{R}^{d_i}, \quad t \in [0,T],$$

$$u^i_t \text{ is nonanticipative measurable w.r.t.} \{y^i(s) : 0 \le s \le t\} \Big\}, \quad \forall i \in \mathbb{Z}_N$$

$\mathbb{A}^i$: closed, bounded and convex, $\forall i \in \mathbb{Z}_N$

$\mathbb{U}^{(N)}[0,T] \triangleq \times_{i=1}^N \mathbb{U}^i[0,T]$

1. **Open Loop,** if $u^i_t = \mu^i(t)$, $t \in [0,T]$, where $\mu^i : [0,T] \longrightarrow \mathbb{A}^i$;
2. **Closed Loop Feedback,** if $u^i_t = \mu^i(t, y^i)$ are nonanticipative functionals of the observation trajectory $y^i(\cdot)$, for $t \in [0,T]$;
3. **Closed Loop Memoryless,** if $u^i_t = \mu^i(t, y^i(t))$, for $t \in [0,T]$.

Team Optimality: Formulation V

Problem

Team Optimality. *Find a* $u^o \in \mathbb{U}^{(N)}[0,T]$ *which satisfies*

$$J(u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \leq J(u^1, u^2, \ldots, u^N), \quad \forall u \in \mathbb{U}^{(N)}[0,T] \tag{6}$$

Person-by-Person (PbP) Optimality. *Find a* $u^o \in \mathbb{U}^{(N)}[0,T]$ *which satisfies*

$$\tilde{J}(u^{i,o}, u^{-i,o}) \leq \tilde{J}(u^i, u^{-i,o}), \quad \forall u^i \in \mathbb{U}^i[0,T], \quad i \in \mathbb{Z}_N \tag{7}$$

$$\tilde{J}(v, u^{-i}) \triangleq J(u^1, u^2, \ldots, u^{i-1}, v, u^{i+1}, \ldots, u^N)$$

Outline

Methodology

- weak variations;
- convexity condition (can be replaced by randomized strategies).
- **Assumptions (A)**
  f is a Borel measurable map $f : [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow \mathbb{R}^n$;
  There exists a $K \in L^{2,+}([0, T], \mathbb{R})$ such that

  (A1) $|f(t, x, u) - f(t, y, u)|_{\mathbb{R}^n} \leq K(t)|x - y|_{\mathbb{R}^n}$ uniformly in $u \in \mathbb{A}^{(N)}$;

  (A2) $|f(t, x, u) - f(t, x, v)|_{\mathbb{R}^n} \leq K(t)|u - v|_{\mathbb{R}^d}$ uniformly in $x \in \mathbb{R}^n$;

  (A3) $|f(t, x, u)|_{\mathbb{R}^n} \leq K(t)(1 + |x|_{\mathbb{R}^n})$ uniformly in $u \in \mathbb{A}^{(N)}$;

  (A4) For any $x, \tilde{x} \in C([0, T], \mathbb{R}^n)$,
  $|h^i(t, x) - h^i(t, \tilde{x})|_{\mathbb{R}^{k_i}} \leq K|x - \tilde{x}|_{C([0,T],\mathbb{R}^n)}$, $K > 0, i = 1, \ldots, N$.

Team Optimality Conditions II

Lemma 1

Suppose Assumptions (A) hold. Then for any $u \in \mathbb{U}^{(N)}[0, T]$, the following hold.

1) The differential system has a unique solution $x \in B^{\infty}([0, T], \mathbb{R}^n)$ which is continuous $x \in C([0, T], \mathbb{R}^n)$;

2) The solutions are continuously dependent on th strategies, in the sense that, as $u^{i,\alpha} \longrightarrow u^{i,o}$ in $\mathbb{U}^i[0, T]$, $\forall i \in \mathbb{Z}_N$, $x^{\alpha} \longrightarrow x^o$ in $B^{\infty}([0, T], \mathbb{R}^n)$.

Team Optimality Conditions III

- **Assumptions (B)**

(B1) The map $f : [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow \mathbb{R}^n$ is continuous in (t, x, u) and continously differentiable with respect to x, u;

(B2) $\{f_x, f_u\}$ are bounded uniformly on $[0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)}$;

(B3) The maps $\ell : [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow (-\infty, \infty]$ is Borel measurable, continuously differentiable with respect to (x, u), the map $\varphi : [0, T] \times \mathbb{R}^n \longrightarrow (-\infty, \infty]$ is continuously differentiable with respect to x, $\ell(t, 0, 0)$ is bounded, and there exist $K_1, K_2 > 0$ such that

$$
\begin{aligned}
|\ell_x(t,x,u)|_{\mathbb{R}^n} + |\ell_u(t,x,u)|_{\mathbb{R}^d} &\leq K_1\big(1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d}\big), \\
|\varphi_x(x)|_{\mathbb{R}^n} &\leq K_2\big(1 + |x|_{\mathbb{R}^n}\big);
\end{aligned}
$$

(B4) $|h^i(t,x)|_{\mathbb{R}^{k_i}} \leq K \sup_{0 \leq s \leq t} \left(1 + |x(s)|^2_{\mathbb{R}^n}\right), \forall t \in [0, T], x \in C([0, T], \mathbb{R}^n), i = 1, \ldots, N.$

Team Optimality Conditions IV

Hamiltonian System

- Hamiltonian:

$$
\begin{aligned}
H :&[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow \mathbb{R}, \\
&H(t,x,\psi,u) \triangleq \langle f(t,x,u), \psi \rangle + \ell(t,x,u). \qquad (8)
\end{aligned}
$$

- Adjoint $\psi \in L^2([0,T], \mathbb{R}^n)$:

$$
\begin{aligned}
\dot{\psi}(t) =& - f_x^*(t, x(t), u_t)\psi(t) - \ell_x(t, x(t), u_t) \\
=& - H_x(t, x(t), \psi(t), u_t), \quad t \in [0,T), \qquad (9) \\
\psi(T) =& \varphi_x(x(T)). \qquad (10)
\end{aligned}
$$

- State:

$$
\begin{aligned}
\dot{x}(t) =& f(t, x(t), u_t) = H_\psi(t, x(t), \psi(t), u_t), \;\; t \in (0,T], \qquad (11) \\
x(0) =& x_0. \qquad (12)
\end{aligned}
$$

Team Optimality Conditions V

Theorem 1. (Necessary conditions of team optimality, ECC2014)
Suppose Assumptions (B) hold, $\mathbb{A}^i \subset \mathbb{R}^{d_i}$ are closed, bounded and convex, and $\{y^i(s) : 0 \leq s \leq t\}$ generates $\mathcal{H}^{y^i}_{0,t}$-a closed subspace of a Hilbert space for $i = 1, \ldots, N$.
For $u^o \in \mathbb{U}^{(N)}[0,T]$ to be team optimal, it is necessary that
(1) There exists a process $\psi^o \in L^2([0,T], \mathbb{R}^n)$;
(2) The triple $\{u^o, x^o, \psi^o\}$ satisfy the inequality:

$$\sum_{i=1}^{N} \int_0^T \langle H_{u^i}(t, x^o(t), \psi^o(t), u^o_t), u^i_t - u^{i,o}_t \rangle dt \geq 0, \ \forall u \in \mathbb{U}^{(N)}[0,T]; \quad (13)$$

(3) $u^o \in \mathbb{U}^{(N)}[0,T]$ satisfies

$$\langle \mathbf{P}_{\mathcal{H}^{y^i}_{0,t}} \Big(H_{u^i}(t, x^o(t), \psi^o(t), u^o_t) \Big), v^i - u^{i,o}_t \rangle \geq 0,$$
$$\forall v^i \in \mathbb{A}^i, \quad t \in [0,T], \quad i = 1, 2, \ldots, N. \quad (14)$$

Team Optimality Conditions VI

Theorem: Sufficient Conditions for Team Optimality

Suppose the conditions of Theorem 1 hold.
Let $(u^o(\cdot), x^o(\cdot))$ denote any control-state pair and let $\psi^o(\cdot)$ the corresponding adjoint variable.
Suppose the following conditions hold.

(C1) $H(t, \cdot, x, u), t \in [0, T]$ is convex in $(x, u) \in \mathbb{R}^n \times \mathbb{A}^{(N)}$;

(C2) $\varphi(\cdot)$ is convex in $x \in \mathbb{R}^n$.

Then

- $(x^o(\cdot), u^o(\cdot))$ is a team optimal pair if it satisfies the conditional Hamiltonian;
- PbP optimality implies team optimality.

Outline

$$
\begin{aligned}
f(t,x,u) =& A(t)x + b(t) + B(t)u,\\
\ell(t,x) =& \frac{1}{2}\langle u, R(t)u\rangle + \frac{1}{2}\langle x, H(t)x\rangle + \langle x, F(t)\rangle + \langle u, E(t)x\rangle + \langle u, m(t)\rangle,\\
\varphi(x) =& \frac{1}{2}\langle x, M(T)x\rangle + \langle x, N(T)\rangle,
\end{aligned}
$$

The projected Hamiltonians give optimal strategies:

$$
\begin{aligned}
u_t^{i,o} = -R_{ii}^{-1}(t)\Big\{ & m^i(t) + \sum_{j=1}^{N} E_{ij}(t)\mathbf{P}_{\mathcal{H}_{0,t}^{y^i}}\Big(x^{j,o}(t)\Big)\\
& + \sum_{j=1, j\neq i}^{N} R_{ij}(t)\mathbf{P}_{\mathcal{H}_{0,t}^{y^i}}\Big(u_t^{j,o}\Big) + B^{(i),*}(t)\mathbf{P}_{\mathcal{H}_{0,t}^{y^i}}\Big(\psi^o(t)\Big)\Big\}, \quad i = 1,2,\ldots,N.
\end{aligned}
$$

Outline

Conclusions & Future Work I

Conclusions.

- Maximum principle + orthogonal projections apply to deterministic decentralized optimization.
- General constraints, i.e., state, control, integral, etc., can be handled.

Future Work

- Compute examples.
- Develop discrete-time & minimax-noncooperative games.

Recent work on decentralized stochastic decision systems

- Charalambous-Ahmed-CDC: 2013, Charalambous-Ahmed-MTNS:2014, Arxiv.

Outline

References I

- Strong formulation + Examples and Applications

1. N. U. Ahmed and C. D. Charalambous, *Stochastic Minimum Principle for Partially Observed Systems Subject to Continuous and Jump Diffusion Processes and Driven by Relaxed Controls*, SIAM Journal on Control and Optimization, pp.3235-3257, 2013.
2. C. D. Charalambous and N. U. Ahmed, *Centralized Versus Decentralized Team Optimality of Distributed Stochastic Differential Decision Systems with Noiseless Information Structures-Part II: Applications*, IEEE Transactions on Automatic Control (submitted), 2013, http://arxiv.org/abs/1302.3416.
3. C. D. Charalambous and N. U. Ahmed , *Team Optimality Conditions of Distributed Stochastic Differential Decision Systems with Decentralized Noisy Information Structures*, IEEE Transactions on Automatic Control (submitted, 2013), http://arxiv.org/abs/1304.3246.

References II

- Weak Girsanov's formulation + examples and applications

1. C. D. Charalambous, *Dynamic Team Theory of Stochastic Differential Decision Systems with Decentralized Noisy Information Structures via Girsanov's Measure Transformation*, MCSS (submitted), 2013, http://arxiv.org/abs/1309.1913.
2. C. D. Charalambous, *Dynamic Team Theory of Stochastic Differential Decision Systems with Decentralized Noiseless Feedback Information Structures via Girsanov's Measure Transformation*, MCSS (submitted), 2013, http://arxiv.org/abs/1310.1488.