Average Consensus in the Presence of Dynamically Changing Directed Topologies and Time Delays

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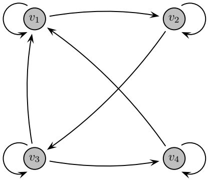
Outline

- Motivation - Introduction
- Notation and mathematical preliminaries
- Average consensus in digraphs
  - Ratio consensus
  - Ratio consensus with time-delays
- **Contribution:** Ratio consensus with dynamically changing directed topologies and time-delays
- Examples
- Concluding remarks and future directions

- Distributed systems conveniently captured by digraphs
  1. Components represented by vertices (nodes)
  2. Communication and sensing links represented by edges



- Consider a network with nodes $(v_1, v_2, \ldots, v_N)$
  (e.g., sensor networks, network of mappers in MapReduce, etc.)
- Nodes can receive information according to (possibly directed) communication links
- Each node v_j has some initial value $x_j[0]$
  (e.g., measurement, position, velocity, etc.)

Consensus and Average Consensus

- *Typical objective:* Calculate functions of initial values in a distributed manner (e.g., $\max_\ell \{x_\ell[0]\}$, $\sum_\ell x_\ell^2[0]$, etc.)

- *Consensus:* All nodes calculate (in a distributed manner, each time using only local information) *same function* of initial values $x_1[0], x_2[0], \ldots, x_N[0]$

- *Average Consensus:* All nodes calculate (in a distributed manner) the *average* $\overline{x} \equiv \frac{1}{N} \sum_{\ell=1}^{N} x_\ell[0]$ (where N is the number of nodes)

- Possible centralized strategy: Route all values to a single entity (leading node) who then determines the function value (e.g., average) and routes it back to the nodes

- Average serves as primitive for estimation, inference and diagnosis (easily adjusted to arbitrary linear functions)

Graph Notation

- *Digraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
  - Nodes (system components) $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$
  - Edges (directed communication links) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ where $(v_j, v_i) \in \mathcal{E}$ iff node v_j can receive information from node v_i
  - In-neighbors $\mathcal{N}_j^- = \{v_i \mid (v_j, v_i) \in \mathcal{E}\}$; in-degree $\mathcal{D}_j^- = |\mathcal{N}_j^-|$
  - Out-neighbors $\mathcal{N}_j^+ = \{v_l \mid (v_l, v_j) \in \mathcal{E}\}$; out-degree $\mathcal{D}_j^+ = |\mathcal{N}_j^+|$
- *Adjacency matrix A:* $A(j,i) = 1$ if $(v_j, v_i) \in \mathcal{E}$; $A(j,i) = 0$ otherwise
- *Undirected graph:* $(v_j, v_i) \in \mathcal{E}$ iff $(v_i, v_j) \in \mathcal{E}$ (bidirectional links)
  In undirected graphs, we have (for each node j)
  $\mathcal{N}_j^+ = \mathcal{N}_j^-$ and $\mathcal{D}_j^+ = \mathcal{D}_j^- = \mathcal{D}_j$; also, $A = A^T$
- *(Strongly) connected (di)graph* if for any $i, j \in \mathcal{V}$, $j \neq i$, there exists a (directed) path connecting them, i.e.,

$$v_i = v_{i_0} \to v_{i_1},\ v_{i_1} \to v_{i_2},\ \ldots,\ v_{i_{t-1}} \to v_{i_t} = v_j$$

- Average Consensus: All nodes calculate (in a distributed manner) the average $\frac{1}{N}\sum_{\ell=1}^{N} x_\ell[0]$ of initial values
- Consider linear iterations of the form

$$x_j[k+1] = p_{jj}x_j[k] + \sum_{i \in \mathcal{N}_j^-} p_{ji}x_i[k] \;,\; \forall j \in \{1, 2, ..., N\}$$

  where p_{ji} are constant weights and $x_j[0]$ is the initial value of node j
- Can be written in compact form as

$$\begin{aligned} x[k+1] &= Px[k] \\ x[0] &= [x_1[0]\; x_2[0]\; \ldots\; x_N[0]]^T \end{aligned}$$

- Weight matrix P such that $P(j,i) = p_{ji}$
  Note: $P(j,i) = 0$ if $(j,i) \notin \mathcal{E}$

Conditions for Asymptotic Average Consensus

- Necessary and sufficient conditions on P for asymptotic average consensus [Xiao & Boyd, 2004]
  1. P has a simple eigenvalue at 1 with left eigenvector $\mathbf{1}^T = [1\ 1\ \dots\ 1]$ and right eigenvector $\mathbf{1} = [1\ 1\ \dots\ 1]^T$
  2. All other eigenvalues of P have magnitude strictly smaller than 1
- As $k \to \infty$, $P^k \to \frac{1}{N}\mathbf{1}\mathbf{1}^T$ which implies that

$$\lim_{k\to\infty} x[k] = \frac{1}{N}\mathbf{1}\mathbf{1}^T x[0] = \left(\frac{\sum_{\ell=1}^{N} x_\ell[0]}{N}\right)\mathbf{1} \equiv \overline{x}\mathbf{1}$$

- Nonnegative $p_{ji} \implies P$ is primitive bistochastic

How to distributively reach the average in digraphs?

Average Consensus in Digraphs

- In undirected graphs nodes have a number of ways to distributively choose their weights so as to form a bistochastic matrix
- Digraphs not as easy to handle, even in a centralized manner (because in general $\mathcal{D}_j^+ \neq \mathcal{D}_j^-$)
- Approaches:
  - Distributed algorithms to obtain weights that form bistochastic matrices [Gharesifard & Cortés, 2012], [Rikos *et al*, 2014]
  - Distributed approaches that introduce additional state variables and use broadcast gossip [Franceschelli *et al*, 2011], [Cai & Ishii, 2011] to reach average consensus *asymptotically*
  - Run two coupled iterations simultaneously (ratio consensus) [Benezit *et al*, 2010], [A.D. Domíngez-García & C.N.H., 2010] that reach the average consensus *asymptotically*

- Run two iterations [Benezit *et al*, 2010], [A.D. Domíngez-García & C.N.H., 2010]

$$
\begin{aligned}
\pi[k+1] &= P_c \pi[k] \\
\pi[0] &= [\pi_1[0] \ldots \pi_N[0]]^T
\end{aligned}
\quad\Bigg|\quad
\begin{aligned}
x[k+1] &= P_c x[k] \\
x[0] &= \mathbf{1}
\end{aligned}
$$

- Matrix P_c st $P_c(l,j) = \frac{1}{1+\mathcal{D}_j^+}$ for $v_l \in \mathcal{N}_j^+$ (zero otherwise)

- Since P_c is primitive column stochastic, we know that as $k \to \infty$, $P_c^k \to \mathbf{v}\mathbf{1}^T$ for a *strictly positive* vector $\mathbf{v}$ such that $\mathbf{v} = P_c\mathbf{v}$ ($\mathbf{v}$ is *normalized* so that its entries sum to unity)

- This implies that

$$
\begin{aligned}
\lim_{k\to\infty} \pi[k] &= \mathbf{v}\mathbf{1}^T \pi[0] = \left(\sum_{\ell=1}^N \pi_\ell[0]\right)\mathbf{v} \\
\lim_{k\to\infty} x[k] &= \mathbf{v}\mathbf{1}^T x[0] = N\mathbf{v}
\end{aligned}
$$

- For all nodes $j \in \{1, 2, \ldots, N\}$, ratio converges

$$
\frac{\pi_j[k]}{x_j[k]} \to \frac{v_j \sum_{\ell=1}^N \pi_\ell[0]}{v_j N} = \frac{\sum_{\ell=1}^N \pi_\ell[0]}{N} \equiv \overline{\pi}
$$

Average Consensus in the Presence of Delays

- A transmission on link (v_j, v_i) (from node v_i to node v_j) at time-step k can be delayed by 0, 1, ..., D steps (arriving at node v_j respectively at time-step $k+1$, $k+2$, ..., $k+D+1$)
- Unknown time-dependent delay function for each link (v_j, v_i)

$$delay_{(j,i)}[k] = d \ , \quad d \in \{0, 1, 2, ..., D\}$$

- $delay_{(j,j)}[k] = 0$ for all k and $v_j \in \mathcal{V}$
(own value is always available)
- For convenience, we use indicator functions

$$I_{k,j,i}(d) = \begin{cases} 1, & \text{if } delay_{(j,i)}[k] = d \\ 0, & \text{otherwise} \end{cases}$$

Can ratio consensus reach average consensus in the presence of time-varying delays?

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Can ratio consensus reach average consensus in the presence of time-varying delays?

- Ratio consensus can be adopted to work in the presence of bounded time-varying delays [C.N.H. & T.C., 2011], [C.N.H. & T.C., 2014]

- Run two delayed iterations [C.N.H. & T.C., 2011], [C.N.H. & T.C., 2014]

$$\begin{aligned} \pi_j[k+1] &= p_{jj}\pi_j[k] + \sum_{i \in \mathcal{N}_j^-} p_{ji} \sum_{d=0}^{D} \pi_i[k-d] I_{k-d,j,i}(d) \\ \pi[0] &= [\pi_1[0] \dots \pi_N[0]]^T \end{aligned}$$

$$\begin{aligned} x_j[k+1] &= p_{jj}x_j[k] + \sum_{i \in \mathcal{N}_j^-} p_{ji} \sum_{d=0}^{D} x_i[k-d] I_{k-d,j,i}(d) \\ x[0] &= \mathbf{1} \end{aligned}$$

- Transmissions from node v_i to node v_j at time step k undergo identical delays in both iterations
(e.g., the pair of values are put in the same packet)
- The ratio $\pi_j[k]/x_j[k]$ converges in the limit to $\frac{\sum_{\ell=1}^{N} \pi_\ell[0]}{N}$ for all nodes $v_j \in \mathcal{V}$

Can ratio consensus also handle time-varying topologies on top of delays?

Average Consensus in the Presence of Delays **and Switchings**

- **Assumptions:**
  1. At each time k, each node v_j knows the number of nodes receiving its message - this assumption can be relaxed
  2. The delays are bounded
  3. There exist paths between any pair of nodes infinitely often

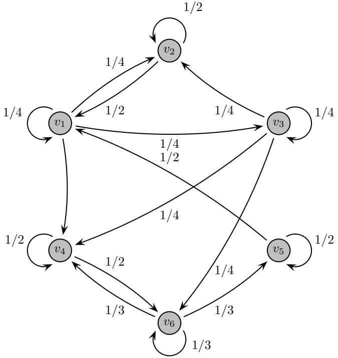
- Run two delayed iterations with time-varying coefficients

$$
\begin{aligned}
\pi_j[k+1] &= p_{jj}[k]\pi_j[k] + \sum_{d=0}^{D} \sum_{v_i \in \mathcal{N}_j^-[k-d]} \pi_{j\leftarrow i}[k-d] I_{k-d,ji}[d] \\
\pi[0] &= [\pi_1[0] \ldots \pi_N[0]]^T
\end{aligned}
$$

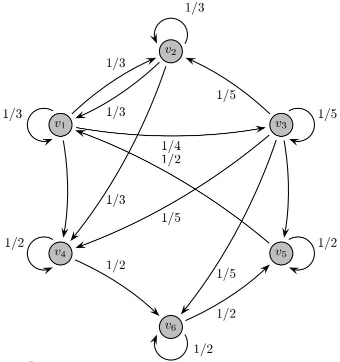
$$
\begin{aligned}
x_j[k+1] &= p_{jj}[k]x_j[k] + \sum_{d=0}^{D} \sum_{v_i \in \mathcal{N}_j^-[k-d]} x_{j\leftarrow i}[k-d] I_{k-d,ji}[d] \\
x[0] &= \mathbf{1}
\end{aligned}
$$

where $x_{j\leftarrow i}[k-d] \triangleq p_{ji}[k-d]x_i[k-d]$ is the value sent from node v_i to node v_j at time step $k-d$ that suffers delay d

- Initial conditions $\pi[0] = (-1 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 3)^T$ and $x[0] = \mathbf{1}$
- Each node v_j chooses its self-weight and the weight of the links to its out-neighbors to be $(1 + \mathcal{D}_j^+[k])^{-1}$

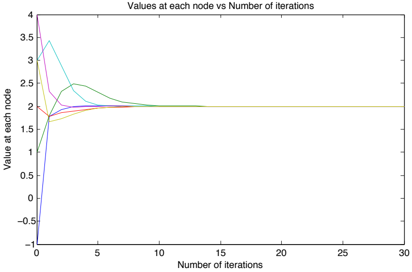


Connections and weights at time-instant k_1

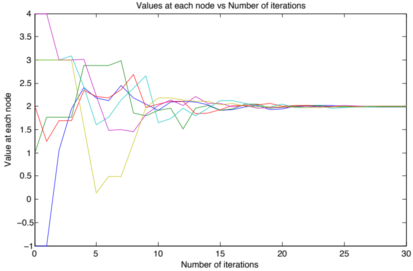


Connections and weights at time-instant k_2

- Iterations converge to the average
- maximum delay $\bar{\tau} = 5$



Switchings, no delays



With switchings and delays

Sketch of proof (1/2)

- To handle delays (see [C.N.H. & T.C., 2014]) we introduce DN "virtual" nodes (for a total of $(D+1)N$ nodes) so that we can write

$$\overline{x}[k+1] = P[k]\overline{x}[k] \ ,$$

where

$$P[k] \triangleq \begin{pmatrix} P_0[k] & I_{N\times N} & 0 & \cdots & 0 \\ P_1[k] & 0 & I_{N\times N} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{D-1}[k] & 0 & 0 & \cdots & I_{N\times N} \\ P_D[k] & 0 & 0 & \cdots & 0 \end{pmatrix},$$

with

$$\overline{x}[k] = \left(x^T[k] \ \ x^{(1)}[k] \ \ldots \ x^{(D)}[k] \right)^T, x^{(d)}[k] = \left(x_1^{(d)}[k] \ \ldots \ x_N^{(d)}[k] \right)$$

- There exists ℓ and an infinite sequence of time instants $t_0, t_1, \ldots, t_m, \ldots$, such that for any $m \in \mathbb{Z}_+$, $0 < t_{m+1} - t_m \leq \ell < \infty$, $m \in \mathbb{Z}_+$, and the union of graphs $\mathcal{G}[t_m], \mathcal{G}[t_m+1], \ldots, \mathcal{G}[t_{m+1}-1]$ is strongly connected (assumption 3)

Sketch of proof (2/2)

- Let $\overline{P}_{t_{m+1}-t_m} \triangleq P[t_{m+1}-1]P[t_{m+1}-2]\dots P[t_m]$
- The union of graphs $\mathcal{G}[t_m], \mathcal{G}[t_m+1], \dots, \mathcal{G}[t_{m+1}-1]$ is strongly connected and each matrix involved in the product has strictly positive elements on the diagonal $\Rightarrow$ $\overline{P}_{t_{m+1}-t_m}$ is Stochastic, Indecomposable and Aperiodic (SIA)
- Products of matrices of the form $\overline{P}_{t_{m+1}-t_m}$ are SIA
- **Wolfowitz theorem:** For any $\epsilon > 0$, there exists a finite integer $\nu(\epsilon) \in \mathbb{N}$, such that a finite word W given by the product of a collection of ν stochastic matrices of the form $\overline{P}_{t_{m+1}-t_m}$ has all of its columns approximately the same
- This implies that

$$\begin{aligned} \lim_{k\to\infty} \pi[k] &= \mathbf{v}[k]\mathbf{1}^T\pi[0] = \left(\sum_{\ell=1}^N \pi_\ell[0]\right)\mathbf{v}[k] \\ \lim_{k\to\infty} x[k] &= \mathbf{v}[k]\mathbf{1}^T x[0] = N\mathbf{v}[k] \end{aligned}$$

- Therefore,

$$\lim_{k\to\infty} \frac{\pi_j[k]}{x_j[k]} = \frac{\left(\sum_{\ell=1}^N \pi_\ell[0]\right)\mathbf{v}_j[k]}{N\mathbf{v}_j[k]} = \frac{\sum_{\ell=1}^N \pi_\ell[0]}{N}$$

Concluding Remarks and Future Directions

Conclusions:

- Proposed a distributed algorithm
  - operates in directed networks
  - has limited local information exchanges
  - converges to the average in the presence of delays and switchings

Future work:

- Investigation of quantized average consensus in directed graphs with switching topologies and delays

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Thank You!

Questions?

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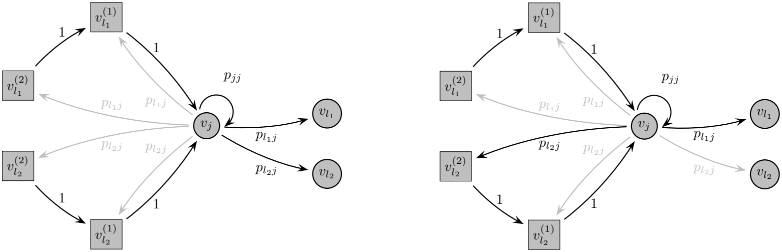
Joker slides

What if a node leave the network?
By an example (1/2)

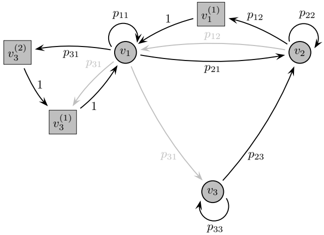
- Max number of time-slots required to receive acknowledgement is 2
- Node v_j directs the weighted messages of the links that no longer exist as delayed information to itself



What if a node leave the network?
By an example (2/2)

Acknowledgement can be:

- via a distress signal
- via a path that reaches v_j



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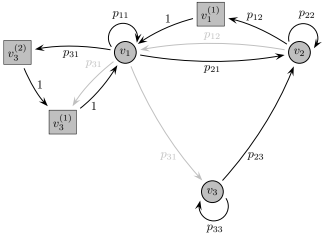

What if communication is not officially terminated?

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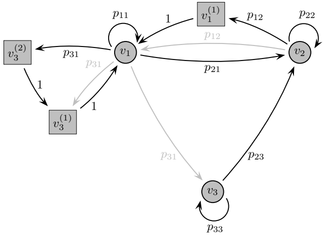
- via a path that reaches v_j



What if communication is not officially terminated?

- Each node v_j broadcasts its own state, as well as the sum of all the values, called the *total mass* in [Vaidya *et al*, 2012]